

On the relevance of the differential expressions $f^2 + f'^2$, $f + f''$ and $ff'' - f'^2$ for the geometrical and mechanical properties of curves

James Bell Cooper

Contents

1	Introduction	4
2	The Kepler problem	9
2.1	Kepler's second law	9
2.2	The inverse square law	10
2.3	The Newton-Somerville equation	11
2.4	A criterion for a power law	13
2.5	Some applications	13
2.6	Two simple cases	14
3	The Affine Curvature of an Orbit Determines the Force Law	16
3.1	Further geometric quantities associated with orbits	16
3.2	Affine curvature	17
3.3	A computational proof	18
3.4	A conceptual proof	19
4	The Kasner-Arnold duality	20
4.1	Sinusoidal spirals and the d -transformation	20
4.2	The general situation	21
5	A Duality between Trajectories for Central and Parallel Force Laws	23
5.1	Curves of the form $(F(t), f(t))$	24
5.2	The duality (quadrality)	25

5.3	Examples of duality	26
5.4	The sinusoidal spirals and catenaries	28
6	On the expressions $f^2 + f'^2$, $f + f''$, $ff'' - f^2$, how to solve the equations $f^2 + f'^2 = af^\alpha$, $f + f'' = bf^\beta$, $ff'' - f^2 = cf^\gamma$ and why one might want to	29
6.1	The relationship between the equations	30
6.2	Some solutions	30
6.3	Where the expressions arise	32
6.4	The calculus of variations	33
6.5	Two more subtle examples	34
6.6	Miscellanea	35
6.6.1	The explicit form of the parametrisation $(F_a(t), f_a(t))$	35
6.6.2	The connection with contact geometry	35
6.6.3	Curves with prescribed curvature	35
6.6.4	The rectification problem	36
6.6.5	The Weingarten mapping of surfaces of revolution . . .	37
6.6.6	Remark	38
7	The trajectories for parallel power laws	38
7.1	Planar motion under a parallel law	40
7.2	Rectilinear motion	45

Abstract

The purpose of this article is to give a *pot-pourri* of results on the mechanical and geometrical properties of curves and explicit solutions to problems on trajectories of particles under suitable force laws. The factor which unifies these rather disparate results is the ubiquity of the expressions in the title. We show how this explains in a unified fashion a plethora of properties of a large class of special curves. We also introduce two related ideas, the so-called d -transformation key res of a function and a new duality between trajectories for central force laws and those for parallel laws. The former will be our principal tool for obtaining many results by a reduction to cases which can be solved by elementary methods; the latter will allow us to move back and forth between results for central force laws and corresponding ones for parallel laws.

Our treatment begins with an overview of the theory of the motion of a planet under a centripetal law, in particular one which varies as a power of the distance from the centre. We derive directly a criterion for a given orbit under a centripetal force to correspond to a power law, This is essentially a modern version of Newton's method.

It is implicit in Newton's treatment of the two-body problem that a single orbit suffices to establish the force law and we give a short and elementary proof that this is true in a strong form—namely that knowledge of the affine curvature on an infinitesimal segment suffices.

We then use the above criterion to give a detailed discussion of the Kasner-Arnold duality between power laws. This involves the first two expressions of our title and the d -transformation mentioned above.

We proceed to discuss two related themes in the differential geometry of special curves—the three differential expressions of the title and a duality between curves which have natural representations as spirals and those which can be conveniently described by parametrisations of the form $(F(t), f(t))$, where F is a primitive of f . We show that the former are ubiquitous in the computation of geometrical or mechanical properties of such curves and display two families, the MacLaurin spirals and catenaries, which are particularly rich in these respects.

In the final section, we give explicit parametrisations of trajectories under a parallel force field, emphasising again the case of power laws, and show that the differential expressions of the title and their stability properties with respect to the d -transformation provide a unified approach. We close with a brief discussion of the theme of recreating curves from their curvature functions.

1 Introduction

We start the technical part of this paper in section 2 below with an overview of one of the most fascinating topics in the history of mathematical physics: the relationship between Kepler’s and Newton’s laws of motion. This was the central theme of Newton’s *Principia*. We are particularly interested in the following two statements about an object (called the planet) which moves around a second one (the sun) subject to a centripetal force (equivalently, in such a manner that Kepler’s second law holds—equal areas are swept out in equal times):

- I. If the orbits are conic sections (more precisely ellipses, parabolas or hyperbolas with the sun at a focus), then the force is inversely proportional to r^2 where r is the distance to the sun;
- II. The converse statement: if the planet moves under an inverse square law, then the orbits are as above.

In the first edition of his monumental treatise [Ne] Newton derived the statements of I (treating the three cases separately) and stated that the converse, i.e., II, holds. In later editions he added a proof of this converse, albeit one which is the subject of some controversy to this day.

Newton’s solution to what is often referred to as the Kepler problem is generally regarded as one of the key moments in the history of science and represented the culmination of the work of Tycho Brahe, who collected the data on the planetary motions, and Kepler, who distilled his three laws from this data.

In addition, Newton developed a criterion which allowed him to deduce from the geometrical form of an observed orbit whether the planet was moving under a power law and to determine which power was involved. He used this to investigate further exotic orbits which satisfy power laws not of the Kepler variety. Two of the more remarkable facts that he deduced were that orbits consisting of circles which pass *through* the sun and conic sections with the sun at the *centre* also arise from power laws (these cases will be dealt with below).

It is implicit in Newton’s work that the geometrical form of a *single* orbit suffices to determine the force law. We shall demonstrate that if we observe an orbit of a planet moving under a central field, then we can deduce the force law from its affine curvature.

Our statements above use (as do all standard treatments of the dynamics of planetary motion) several hidden assumptions, since from the observation of a single orbit we can, in the most general situation, clearly only deduce information about the force law at the points through which the planet passes. Basically, we are assuming, in addition to Kepler’s area law (equivalently,

that the motion is determined by a central force emanating from the sun):

- a) that the force is independent of time;
- b) that the force depends only on the distance from its source (the sun) , i.e., it is invariant under rotation around the sun.
- c) that the force depends on the radius in a (real) analytic fashion. Hence, if we observe a non-trivial orbit (that is, one which is not a circle with centre at the sun) this will determine the nature of the analytic function on a non-degenerate interval and so in its entirety. (In our statements below we shall always assume that we do not have such a circular orbit in order to avoid this case—*any* force law which satisfies a), b) and c) clearly has circular orbits as special solutions.)

Note that if we assume c) then it is not necessary to observe a complete orbit to derive the force law. Any non-trivial segment will suffice.

One of our central results can be expressed succinctly as follows:

- A) The force is proportional to a power of the distance from the central point (the “sun”) if and only if the affine curvature κ_{aff} is also proportional to a power of this distance. In addition, the appropriate powers are related in a simple fashion, namely $K \propto \frac{1}{r^\alpha} \Leftrightarrow \kappa_{\text{aff}} \propto \frac{1}{r^{\alpha+1}}$.
- B) More generally, we have a force law $K \propto \phi(r)$ if and only if $\kappa_{\text{aff}} \propto \phi(r)/r$ where ϕ is a suitably smooth function (we use the proportionality sign rather than equality to avoid inserting constants).

Curiously, we have been unable to find any relation between force laws and affine curvature in the literature (with one exception—see below). It is known that Newton used what in modern terms would be called curvature arguments but he can hardly have used affine curvature since the concept was only invented in the last century (see [Sc2] for a history of affine differential geometry). There are various articles on Newton’s use of curvature (see, for example, [Br], [Co2]) but they clearly refer to the classical (Euclidean) curvature which was available to him as the inverse of the radius of the osculating circle.

There is, however, one result on celestial mechanics which does invoke (implicitly) Euclidean curvature, namely the theorem of Hamilton [Ha] which states that the motion is Keplerian , i.e., corresponds to an inverse square law $K \propto \frac{1}{r^2}$ if and only if the hodograph is a circle , i.e., has constant curvature. (Recall that the hodograph of the motion is the curve traced out by the velocity vector).

We can incorporate this into our scheme as follows:

- C) The motion corresponds to a power law if and only if the Euclidean curvature κ_{hod} of the hodograph is proportional to a power of the distance from the corresponding planetary position to the origin. More precisely, we have $K \propto r^\alpha$ if and only if $\kappa_{\text{hod}} \propto r^{-\alpha-2}$.

The relationship between the force law and the two curvatures can be expressed succinctly in the formulae:

$$\kappa_{\text{hod}} \cdot \kappa_{\text{aff}} \propto \frac{1}{r^3}, \quad \kappa_{\text{hod}} \cdot K \propto \frac{1}{r^2}, \quad \frac{K}{\kappa_{\text{aff}}} \propto r.$$

In the course of our investigations we stumbled in a natural manner on the three-parameter family of functions

$$f(t) = p(\cos(d(t - t_0)))^{\frac{1}{d}}$$

which have the property that the corresponding orbits $rf(\theta) = 1$ are induced by power laws. The crucial fact is that the members of this family have the special property that the result of an application of any of the differential expressions of our title leads to a function which is proportional to a power of f ; this fact will be the key to our treatment. More precisely, if

$$f(t) = p(\cos d(t - t_0))^{\frac{1}{d}}$$

then

$$f^2 + f'^2 = p^2 f^{-2d+2}, \quad f + f'' = -p^2(d-1)f^{-2d+1}, \quad ff'' - f'^2 = -p^2 df^{-2d+2}.$$

Thus the curves of the form $r^d \cos(d\theta) = 1$ all describe planetary orbits which are induced by power laws (we omit the p and t_0 since these correspond to the simple geometrical operations of dilation and rotation respectively). These curves were introduced by the renowned mathematician Colin MacLaurin, who was aware of precisely this fact. Further literature searches revealed that they had been investigated thoroughly in the period of classical differential geometry (geometry of special curves). Their equations are usually written in the form $r^n = \sin(n\theta)$ and they are known under the name of the MacLaurin spirals (sometimes sinusoidal spirals). A central thesis of our paper is that these all arise from the simple case $f(t) = p \cos(t - t_0)$ (a straight line) under what we call the d -transformation which associates to a function f a new one f_d where $f_d(t) = (f(dt))^{\frac{1}{d}}$. The latter's stability properties with respect to the differential expressions of our title explain in a unified manner many of these results.

The two standard treatises on special curves ([Go], [Lo]) devote extensive chapters to the development of some of the remarkable geometrical and mechanical properties of the MacLaurin spirals.

A further theme of our treatment is a duality between the curves discussed so far and a second family of curves which have analogous properties with respect to a force parallel to the y -axis, in particular one which is proportional to a power of the distance to the x -axis.

Three motivating examples are

1. The cycloid $(t - \cos t, 1 + \sin t)$ which is a brachistochrone and tautochrone for a constant force parallel to the y -axis, and a trajectory for a $\frac{1}{y^2}$ law.
2. The circle $(-\cos t, \sin t)$, which is a geodesic for the Poincaré half-plane.
3. the catenary $y = \cosh x$, which is the form taken on by a hanging chain.

We remark that each of these curves can be parametrised in the form $(F(t), f(t))$ where F is a primitive of f which is in turn a function of the above form, i.e., $p(\cos(d(t - t_0)))^{1/d}$ (this is obvious in the first two examples, the third one is more subtle).

The connection to the first class of curves is that there is a deep analogy between the properties of curves with parametrisations $(F(t), f(t))$ (where F is a primitive of f) and those of the spirals $rf(\theta) = 1$. Both have remarkable properties when f has the special form used in the definition of the MacLaurin spirals. It will be one of the main tasks of this article to display the reasons for these two facts.

We shall also be interested in other families of curves associated with a force law—brachistochrones, tautochrones or isochrones, catenaries, elastica, geodesics for suitable metric tensors, and for light rays in media where the index of refraction is proportional to a power of the distance to a central point. For the sake of conciseness we will simply refer to these curves as *trajectories*.

As a sample of these properties we mention that the MacLaurin spiral $r^d = \cos(d\theta)$:

1. satisfies $\kappa \propto \frac{1}{r^{d-1}}$ (κ is the curvature);
2. is an orbit for $K \propto \frac{1}{r^{2d+3}}$;
3. is a brachistochrone for $K \propto \frac{1}{r^{2d+1}}$;
4. is a catenary for $K \propto \frac{1}{r^{d+2}}$.

(see [Go]).

3) and 4) are examples of solutions of problems of the type: minimise $\int r^\alpha ds$ for a suitable index α . This suggests that the functions of the form $p(\cos(d(t - t_0)))^{\frac{1}{\alpha}}$ will supply solutions to many concrete problems of the calculus of variations, a fact which we shall verify and explain from our unified point of view.

Our last theme is a description of the trajectories of particles moving under parallel-force laws. Recall that the trajectories of a particle moving under a given force law are the solution curves of the differential equations

$$\frac{d^2x}{dt^2} = f(x(t), y(t)) \quad \frac{d^2y}{dt^2} = g(x(t), y(t)),$$

where $(f(x, y), g(x, y))$ is the force at the point (x, y) . The most interesting cases are, of course, that of a central force, i.e., one of the form $f(r)(\cos \theta, \sin \theta)$ (where r and θ are polar coordinates) which we discuss in section 2, or a parallel force of the form $(0, f(y))$. Of particular interest are the cases where f is a power function, i.e., $K \propto r^\alpha$ in the central case or $K \propto y^\alpha$ in the parallel one. Note that the family of trajectories is unchanged if we multiply the force field by a *positive* constant. Hence in the above cases, only the sign of the constant of proportionality affects the family of trajectories, not its absolute value.

Of course, the famous result of Galilei (the trajectories for a constant parallel force are parabolas) is, together with those of Newton, one of the key results in the history of physics.

One of Newton's less well-known discoveries is the fact that a Dido circle (i.e., one which is perpendicular to the x -axis) is a trajectory for a y^{-3} law. This will follow from our results; but we can go much further than Newton here. Thus, we obtain *all* trajectories for such a law and can show that these curves are the only *circles* which are trajectories for a y^α law.

The study of trajectories generated by force laws was an area of very active research in the first half of the previous century and is associated with Edward Kasner and his students. These investigators were interested in general properties of families of curves which arise as such trajectories, rather than in the explicit form of the families for concrete laws. We think that it is of some interest to document the fact that we can write down explicitly *all* the trajectories, using simple elementary functions (and an integration), in the case of a parallel power law.

It is interesting that for the better-known case of a central power law, there are just three indices for which all of the trajectories can be described explicitly using elementary functions: the Kepler case $K \propto r^{-2}$, Hooke's law

$K \propto r$ (where the orbits are conic sections with centre at the origin) and the Cotes' spirals ($K \propto r^{-3}$) which we discuss in section 2. In general, the trajectories for a given force law form a three-parameter or ∞^3 family. For the general central power law the MacLaurin spirals mentioned above provide an explicit ∞^2 family of trajectories, but in the general case the remaining ones can as far as we know only be described indirectly as far as we know (using functions which can be determined implicitly after a quadrature). In the Kepler case for instance, MacLaurin's family only picks up the parabolic orbits.

The special case of rectilinear motion turns out, perhaps surprisingly, to be more intricate, and it is interesting to note that Newton, in his *Principia*, devoted a whole section to this case, which he regarded as a limiting case of the planar one (for a central force). In this case, it is, of course, not the geometrical form of the motion which is of interest but its direct description, i.e. formulae for the position as a function of time. Here the results are less satisfactory in the sense that we have to use not just elementary functions but also the inverse of such a function.

In the final section we give descriptions of curves which satisfy the condition that their curvature is proportional to a power of the distance from the x -axis.

Many of the results of this article are, of course, known; our contribution has been to provide a unified approach. However, we do give explicit formulae and introduce special curves.

As mentioned in the text, we were led to consider the questions below during the course of a cooperation with T. Russell and the late P.A. Samuelson to whom we owe thanks for many fruitful discussions. We would also like to thank Iain Fraser who read through and commented on an earlier version of the text (and, in particular, for his suggesting the word “quadrality” as the appropriate substitute for “duality” for foursomes).

We now turn to the technical part of this article. We begin with a survey of the Kepler problem.

2 The Kepler problem

2.1 Kepler's second law

In this section we prove the basic fact that Kepler's second law (i.e., that the area swept out by the planet in a given interval of time is constant) is equivalent to the fact that the force is centripetal. This well-known fact was proved by Newton in [Ne], but since it is central to our approach we give a

proof in the spirit of what follows.

We assume that the orbit of the planet has polar form $r = f(\theta)$ for a smooth non-negative function f . Usually f will be strictly positive, but in at least one case we will allow it to have a zero (i.e. for the orbit to pass through the origin). Note that we are not assuming that f is periodic, i.e. that the orbits are closed.

In this situation, the motion is determined when we know θ as a function of time. The equations of motion are then

$$x(t) = f(\theta(t)) \cos \theta(t), \quad y(t) = f(\theta(t)) \sin \theta(t).$$

The area swept out in the interval from t_0 to t is

$$A(t) = 1/2 \int_{\theta(t_0)}^{\theta(t)} f^2(u) du.$$

The component of the acceleration perpendicular to the unit vector

$$(\cos \theta(t), \sin \theta(t))$$

is the scalar product of the vector $(-\sin \theta(t), \cos \theta(t))$ with the second derivative of $(f(\theta(t))(\cos \theta(t), \sin \theta(t)))$ and an elementary calculation shows that this is

$$2f'(\theta(t))\theta'^2(t) + f(\theta(t))\theta''(t).$$

On the other hand, we can use the fundamental theorem of calculus and the chain rule to see that

$$A''(t) = f(\theta(t))f'(\theta(t))\theta'(t)^2 + \frac{1}{2}f^2(\theta(t))\theta''(t).$$

Hence $A''(t)$ is a multiple of the component of the acceleration and so the vanishing of $A''(t)$ (which is just the analytical expression of Kepler's second law) is equivalent to the vanishing of the force component perpendicular to the vector from the sun to the planet.

2.2 The inverse square law

In order to prepare the reader for what follows, we consider briefly the Kepler problem in its original form, i.e., for the inverse square law. We shall show shortly that if we write the equation of an orbit in the rather unusual form $rf(\theta) = 1$, then the force is proportional to a power of the distance if and only if f satisfies a differential equation of the form

$$f(\theta) + f''(\theta) = cf^\alpha(\theta)$$

for some constant c and index α . Then $K \propto r^{-2-\alpha}$. (The left hand side of this equation is, of course, the first of our differential expressions). Thus an inverse square law corresponds to the case where $\alpha = 0$ and this gives a clue to why the Kepler universe is particularly stable: the above equation is then linear, in fact it is simply $f(\theta) + f''(\theta) = c$. Of course this can be solved immediately, and the reader will see that the result is precisely that of Newton. Since the argument works in both directions, we have thus shown the equivalence of I and II above, i.e., completed the Newtonian programme with respect to the inverse square law. Of course, we still have to derive the above equation and this we will now do.

2.3 The Newton-Somerville equation

Suppose that we observe one planetary orbit, which we now write as a polar equation $rf(\theta) = 1$. The reason for using this form rather than the more usual one $r = f(\theta)$ employed above is that this leads to a significant increase in transparency and clarity in the computations. In the Keplerian case, where the orbit is a conic section with the sun at a focus, the equation is—for a suitable choice of coordinate system— $r(1 + e \cos \theta) = 1$ where e is the eccentricity. In particular, $0 < e < 1$ corresponds to an elliptical orbit, $e = 1$ to a parabolic, $e > 1$ to a hyperbolic. The fact that $f(\theta) = 1 + e \cos \theta$ is a much simpler function than its reciprocal is a further clue as to why our choice of equation is more natural in this context.

Consider now the configuration consisting of the dilations of the single orbit and the rays going through the origin, i.e., the level curves of the functions (in polar coordinates) $u = rf(\theta), v = \theta$.

This is a so-called S -configuration (see [Co1]) and it is easy to calculate that the mapping $(x, y) \mapsto (U, V)$ where

$$U = \frac{r^2}{2}f(\theta)^2 \text{ and } V = g(\theta)$$

is area-preserving and has our two systems as level curves, where $g(\theta)$ is a primitive of $\frac{1}{f(\theta)^2}$.

It follows that if the area condition is satisfied, then (up to constants) the time is given by $t = g(\theta)$ and we can use this to get θ as a function of t by inverting g . In the interesting examples (e.g. in the Kepler case), one can compute g explicitly, but not its inverse. Fortunately, as we shall see, for our purposes we shall not require this explicit representation for θ as a function of time.

We have gone into this argument in some detail since it displays the connection with the concept of an S -configuration. Of course, Kepler's area condition leads directly to this expression for t .

We can now compute a simple formula for $\frac{dt}{d\theta}$ and so (by the inverse function theorem) for $\frac{d\theta}{dt}$ in terms of f and its derivatives. In fact, as the reader will easily check,

$$\frac{d\theta(t)}{dt} = f^2(\theta(t)).$$

The motion is now (in Cartesian coordinates):

$$x(t) = \frac{\cos \theta(t)}{f(\theta(t))}, \quad y(t) = \frac{\sin \theta(t)}{f(\theta(t))},$$

where $\theta(t) = g^{-1}(t)$.

We now have the machinery we need to compute the derivatives of $(x(t), y(t))$. We differentiate the vector function $(x(t), y(t))$ and use the chain rule to get the velocity vector as a function of time (in terms of f and its derivatives). The result is

$$v(t) = -(\sin \theta(t)f(\theta(t)) + \cos \theta(t)f'(\theta(t)), -\cos \theta(t)f(\theta(t)) + \sin \theta(t)f'(\theta(t))).$$

Similarly, the acceleration vector is

$$a(t) = -(\cos \theta(t), \sin \theta(t))f^2(\theta(t))(f(\theta(t)) + f''(\theta(t))).$$

We can immediately read off from the expression for the acceleration (which is, of course, proportional to the force) that we have a power law if and only if f satisfies the above equation

$$f(\theta) + f''(\theta) = cf^\alpha(\theta).$$

(If we are only interested in the Kepler case, we can stop here since this closes the gap in our considerations above). This equation is equivalent to one which can be found in [Ne]—see [Cha] for a detailed discussion. Because of the central role that it will play in our considerations and of the relevance of the constants, we denote it by $\text{ns}(c, \alpha)$ and call it the Newton-Somerville equation (the first statement in a modern form of this criterion which we have been able to trace is in [So]). A recent reference is [Po]—c.f. equation (6.4) there.

2.4 A criterion for a power law

If we differentiate this equation we get

$$f'(\theta) + f'''(\theta) = c\alpha f^{\alpha-1}(\theta) f'(\theta)$$

and so we can eliminate c by division to get

$$\frac{f'(\theta) + f'''(\theta)}{f(\theta) + f''(\theta)} = \alpha \frac{f'(\theta)}{f(\theta)}$$

, i.e.,

$$\frac{f(\theta)(f'(\theta) + f'''(\theta))}{f'(\theta)(f(\theta) + f''(\theta))} = \alpha.$$

We can reverse this reasoning to deduce that we have a power law if and only if the derivative of the expression

$$\frac{f(f''' + f')}{f'(f + f'')}$$

vanishes, in which case $F \propto r^\beta$, where $\beta = -2 - \alpha$ and α is the (constant) value of $\frac{f(f''' + f')}{f'(f + f'')}$.

The equation for the existence of a power law is thus

$$f'(f + f'')(ff'' + f'^2 + f'f''') + ff'''' - (ff' + ff''')(f'(f' + f''') - f''(f + f'')) = 0.$$

Summarising, the force satisfies a power law if and only if f is a solution of this ODE and the power β is then $-2 - \frac{f(f''' + f')}{f'(f + f'')}$.

Due to the possibility of the functions in the denominator having zeroes, one should perhaps regard this equation more as a heuristic principle. Thus for a given f one tests (e.g. with Mathematica) whether it is a solution of the above equation. If this is the case, one computes the appropriate quotient and verifies it for constancy. Possible zeroes of f' or $f + f''$ can then be investigated with *ad hoc* methods.

2.5 Some applications

Using this machinery we can instantly check whether a given orbit corresponds to a power law. We used it to check all of the examples in Newton. We then introduced parameters into the equations of such orbits and experimented to find combinations which produce further examples. We mention two simple ones:

EXAMPLE We used the above equations to determine when a circular orbit derives from a power law by testing the case of a circle with centre at $(a, 0)$ and radius 1. We found that this satisfies a power law if and only if $a = 0$ or $a = \pm 1$. Hence a circular orbit corresponds to a power law if and only if the sun lies on the circle or at the centre. Both of these cases were discussed in Newton's Principia, but we have found no indication that he knew that these were the only ones. This fact is of some historical interest since one of the hypotheses which Kepler considered and rejected was that the orbit of Mars *was* circular, but with the sun displaced from the centre.

Curiously, when we tried to extend this to the case of ellipses, i.e., to find out if there are other possible positions of the sun (other than the known ones at the centre or at a focus) for a power law, the computations turned out to be too complicated to be completed by Mathematica.

Of the examples which we computed, we mention one which had interesting consequences. We tested the orbit

$$f(\theta) = (a + b \cos(d\theta))^{1/d}$$

for a power law. The above expression was cobbled together out of such examples as $f(\theta) = 1 + \cos \theta$, $f(\theta) = (1 + \cos \theta)^{-1}$, $f(\theta) = \cos \theta$, $f(\theta) = (\cos \theta)^{-1}$. The reason for the unusual nature of the dependency on d will become clear later. Apart from known or trivial cases, this provided us with two families of suitable functions, namely

$$f_d(\theta) = (\cos(d\theta))^{1/d}$$

and

$$g_d(\theta) = (1 + \cos(d\theta))^{1/d}$$

The family

$$f_d(\theta) = (\cos(d\theta))^{1/d}$$

is well known in the classical theory of curves and, as mentioned in the introduction, will play a crucial role in our treatment. Its members satisfy the power law $F \propto r^{-3+2d}$. The second family $g_d(\theta) = (1 + \cos(d\theta))^{1/d}$ coincides essentially with the first one because of the simple identity $(1 + \cos d\theta) = 2 \cos^2(\frac{d\theta}{2})$.

2.6 Two simple cases

We return to our central ODE

$$f + f'' = cf^\alpha.$$

with constants α and c . In general, this equation is non-linear.

However, there are two cases where it *is* linear, namely $\alpha = 0$ (the Kepler case) and $\alpha = 1$ (the $K \propto r^{-3}$ case), but it is linear for different reasons, this is evident from the different role of the constant, and it is instructive to compare the solutions.

$\alpha = 0$: the equation is $f + f'' = c$. This is linear but inhomogeneous, and c plays the role of the inhomogeneous term. This equation can be solved by elementary methods and the solutions are exactly of the form

$$f(\theta) = c + a \cos(\theta) + b \sin(\theta).$$

(This completes our analysis of the Kepler case since these correspond to the polar equations of conic sections with focus at the origin). More precisely, if $c > 0$ these f describe ellipses, parabolas or hyperbolas with the origin as a focus, and the freedom in the choice of the constants a and b (together with dilations) means that a curve specified by a solution of the differential equation, i.e., a curve of the form $r f(\theta) = p$ where f is a solution of the differential equation and p is a scaling factor, is a curve of the required type. For the remaining values of c we get uniform motion in a straight line ($c = 0$), or hyperbolas (the Coulomb case $c < 0$).

There is, however, one more case where this dependence can be computed, namely the case $\alpha = 1$. Then the equation is $f + f'' = cf$ or $f'' = (1 - c)f$. Once again this is linear, but c plays an entirely different role, namely in the coefficient of f . The solutions can be computed very easily and are logarithmic spirals, hyperbolic spirals and epispirals, depending on whether $c < 1$, $c = 1$ or $c > 1$.

In this case we see that the class of orbits defined by the force law depends in an essential manner on the constant of proportionality. This case was settled by Cotes and the spirals which arise are known as Cotes' spirals. For a modern treatment, see Whittaker [Wh].

As a final remark in this section we mention briefly the more general case of a force law which is a sum of two powers (also of considerable historical interest). These will appear again in the next section.

EXAMPLE : Consider the equation

$$f + f'' = af^\alpha + bf^\beta$$

which corresponds to a force law of the form

$$F = \frac{a}{r^{2+\alpha}} + \frac{b}{r^{2+\beta}}.$$

The case $\alpha = 0$ and $\beta = 1$ is interesting both for physical reasons and also because it is the only genuine case of a sum of two powers which can be computed with ease.

In this case the differential equation is $f + f'' = a + bf$ i.e. $f'' = (b - 1)f + a$.

We distinguish the cases:

a) $b < 1$. The solution is

$$\frac{a}{1 - b} + Ae^{(1-b)^{1/2}\theta} + Be^{-(1-b)^{1/2}\theta};$$

b) $b > 1$. The solution is

$$\frac{a}{1 - b} + A \cos[(b - 1)^{1/2}\theta] + B \sin[-(b - 1)^{1/2}\theta];$$

c) $b = 1$. The solution is

$$\frac{a\theta^2}{2} + \theta + B.$$

3 The Affine Curvature of an Orbit Determines the Force Law

It follows from the formulae given above that the statement of the title of this section holds. Due to its intrinsic interest, we go into this in more detail.

3.1 Further geometric quantities associated with orbits

There are two further quantities which are determined by the geometry of the orbit and which turn out to be relevant—these are the curvature functions κ and κ_h of the orbit and of the so-called hodograph, i.e., the curve traced out by the velocity vector. The latter was introduced by Hamilton, who showed that the presence of an inverse square law is equivalent to the fact that the hodograph describes a circle, i.e., a curve with constant curvature. These curvatures can be calculated from the above equations using the standard formulae from differential geometry for the curvature of a parametrised curve. Since one has to differentiate the equations of motion three times to achieve this task, one obtains potentially highly complicated expressions. However, a miracle takes place and they simplify to the tractable and significant formulae:

$$\kappa = \frac{f^3(f + f'')}{(f^2 + f'^2)^{3/2}}, \quad \kappa_h = \frac{1}{f + f''}.$$

(These formulae can be computed by hand—we also checked them using Mathematica). The equation $f + f'' = \frac{1}{\kappa_h}$ can be regarded as a quantitative version of Hamilton’s characterisation of the Kepler case, since the circular form of the hodograph (i.e. the constancy of κ_h) is equivalent to the validity of the equation $f + f'' = c$ for f (we are tacitly assuming that $f + f''$ is positive).

Thus the expression $f(\theta) + f''(\theta)$ which occurs in the central ODE is the radius of curvature of the hodograph. Since this expression and the related one $f^2 + f'^2$ will be of crucial importance below, it is of interest that they can be expressed in terms of the curvatures κ and κ_h , together with f , explicitly. In fact,

$$f^2 + f'^2 = \frac{f^2}{(\kappa\kappa_h)^{2/3}}, \quad f + f'' = \frac{1}{\kappa_h}.$$

Thus we see how two of the expressions from our title arise in a natural way. We now consider the affine curvature of the orbit.

3.2 Affine curvature

Within the context of affine geometry, there are four concepts of curvature (affine curvature, equi-affine curvature, central affine curvature and central equi-affine curvature) depending on which type of geometry is involved. These are characterised by the choice of Lie group to define the geometry in question (in the spirit of Klein’s Erlangen programme)—the affine group, the equi-affine group, the central affine group or the central equi-affine group. For completeness we recall briefly the definitions of these groups.

The affine group is the six-parameter group of all affine transformations of the plane. (It will be convenient to use the classical terminology and refer to this as an ∞^6 -group).

The central affine group is the ∞^4 subgroup of those affine transformations which leave a given point S invariant (S for “sun”)

The equi-affine group is the ∞^5 group consisting of those affine transformations which are area-preserving.

The definition of the central equi-affine group (an ∞^3 -group) should now be self-explanatory.

Each of these groups is associated with an appropriate notion of curvature. In view of Kepler’s second law it is natural to use the last group in dealing with orbital mechanics. In order to avoid the unwieldy terminology “central equi-affine curvature” we will refer to this simply as the affine curvature and denote it by κ_{aff} .

We can give our main result a more intuitive content by recalling that the affine curvature has the following direct geometric interpretation. We denote by S the origin, by P a typical point on the curve and by P' a neighbouring point. We let the tangent to the curve at P meet the ray SP' at Q . Then the affine curvature at P is twice the limit, as P' tends to P , of the quotient of the area of the triangle $P'PQ$ by the cube of that of SPP' (see [Sc] for details).

3.3 A computational proof

We shall begin with a purely computational proof of the relationship between the force law and the affine curvature, since this requires no knowledge of affine geometry apart from the formula for κ_{aff} . The formulae which we develop will also allow us to compute some simple illustrative examples. We will bring a conceptual proof at the end of the article.

The required formula is

$$\kappa_{\text{aff}} = \dot{\mathbf{x}} \wedge \ddot{\mathbf{x}} / (\mathbf{x} \wedge \dot{\mathbf{x}})^3$$

where $\mathbf{x}(t)$ is a parametrisation of the curve and we use the Newtonian dots to indicate differentiation. The wedge product $\mathbf{x} \wedge \mathbf{y}$ of vectors $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ is the determinant $x_1 y_2 - x_2 y_1$ of the corresponding 2×2 matrix (see [6] for the above formula).

If we plug the parametrisation $(x(\theta), y(\theta)) = \frac{(\cos(\theta), \sin(\theta))}{f(\theta)}$ into the formula for κ_{aff} , a simple computation leads to the expression

$$\kappa_{\text{aff}} = (f(\theta) + f''(\theta))f^3(\theta).$$

The curvature of the hodograph is $\frac{1}{f(\theta) + f''(\theta)}$, as can easily be computed by using the following parametrisation, which we include since it is of independent interest (it allows us to compute the hodograph of any planetary orbit from its geometric form):

$$(-f(\theta) \sin \theta - f'(\theta) \cos \theta, f(\theta) \cos \theta - f'(\theta) \sin \theta).$$

We remark that this is the image of $(f'(\theta), f(\theta))$ under the reflection matrix

$$-\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}.$$

We remark that the MacLaurin spirals provide a large class of curves whose affine curvature is proportional to a power of the distance from the origin and which are therefore orbits for power laws.

Remarkably, they also have the property that the *Euclidean* curvature is proportional to a power of the distance to the origin—a fact that follows from the above relationship and the formula for the curvature of the orbit.

Our results establish a relationship between the orbits of power laws and classes of curves characterised by the fact that the affine curvature is proportional to a power of the distance to the origin. There are three cases where one can give simple explicit solutions to the first problem and hence to the second one. These are the Kepler case of an inverse square law, the Hooke case ($K \propto r$) and the case where $K \propto \frac{1}{r^3}$. As noted above, the second case is a well-known result, characterising the curves with constant affine curvature as the conics with centres at the origin. We have been unable to trace the other two in the literature and so state them here for the record. The conics with foci at the origin are characterised by the condition $\kappa_{\text{aff}} \propto r^{-3}$. The third family are the so-called Cotes' spirals, which are characterised by the fact that their affine curvature is proportional to r^{-4} . They have the representation $rf(\theta) = 1$ in polar coordinates, where f is a solution of an equation $f(\theta) + f''(\theta) = cf(\theta)$ for some c . The form of the solution depends on whether $c > 1$, $c = 1$ or $c < 1$. Typical examples are the spirals $r \cos(a\theta) = 1$, $r\theta = 1$, and logarithmic spirals.

3.4 A conceptual proof

We round off this section with a conceptual proof of the result on the affine curvature. As is to be expected this is very simple and follows more or less directly from the definition of the curvature notion that we use (which makes it all the more surprising that this result has not been documented already). Recall that the classical ,i.e., Euclidean, curvature of a curve is defined as follows. One introduces a special parametrisation, by arc-length, and shows that the second derivative of the parametrised curve is then proportional to the normal to the curve at the given point. The curvature is then defined to be the corresponding proportionality factor. In the case of the affine curvature one uses, as special parametrisation, the area spanned by the ray from the origin to a given point P on the curve. In view of Kepler's second law, this parametrisation is proportional to time in the case of a planetary orbit. The second derivative of the parametrisation is then parallel to the ray OP . The affine curvature is defined to be the corresponding proportionality factor (once again, the precise argument and formulae can be found in [6]—see §13). Since the second derivative is proportional to the acceleration, this explains the formulae used in our result.

4 The Kasner-Arnold duality

4.1 Sinusoidal spirals and the d -transformation

We now explain the curious property of the sinusoidal spirals which we noted above, namely that they *all* arise from power laws. We begin with some remarks on these curves. They are usually specified by their polar equations

$$r^n = \cos n\theta \quad \text{or} \quad r^n = \sin n\theta$$

where n is a parameter which can range over the real numbers, the second version explaining the nomenclature. The first version will be more convenient for our purposes (of course, the two are related by a rotation).

This family was introduced by MacLaurin; particular choices of n produce some familiar classical curves. For example, $n = 2$ is a hyperbola (with centre at the origin), $n = -1$ is a line, $n = -\frac{1}{2}$ a parabola (focus at the origin), $n = -\frac{1}{3}$ is Tschirnhausen's cubic (also called Catalan's trisectrix or L'Hospital's cubic), $n = \frac{1}{3}$ Cayley's sextic, $n = \frac{1}{2}$ is the cardioid and $n = 2$ the lemniscate.

In a certain sense the family is generated by the two special cases $n = \pm 1$, i.e., the curves $r = \cos \theta$ and $r = \frac{1}{\cos \theta}$ (a unit circle tangential to the y -axis at the origin and the line $x = 1$ respectively). Since this fact is of some moment in what follows we explain the process. If we employ the substitutions $R = r^n$, $\Theta = n\theta$, then the equation $R = \cos \Theta$ reduces to that of the generic sinusoidal spiral. Now the point with polar coordinates (R, Θ) is the image of (r, θ) under the mapping $z \mapsto z^n$ of the complex plane. In other words, the sinusoidal spirals are the pre-images of the circle $R = \cos \Theta$ or the line $R \cos \Theta = 1$ (depending on the sign of n) under this mapping (alternatively the images under $z \mapsto z^{1/n}$).

At this point we remark that the correct setting for this discussion is not the punctured plane (i.e., \mathbf{R}^2 or \mathbf{C} without the origin) but its universal covering surface, i.e., the Riemann surface of the logarithmic function. This avoids the usual difficulties with the non-uniqueness or discontinuity of the argument function. However, in order to keep our discussion elementary, we shall ignore this subtlety.

Since we have written our curves in the form $rf(\theta) = 1$ (rather than $r = g(\theta)$ which is more natural in the context of curve theory), we shall replace n by $-d$ in the above equations, which now take the forms

$$r(\cos d\theta)^{1/d} = 1$$

as before.

These considerations lead naturally to the concept of the *d-transformation* f_d of a function f , where

$$f_d(t) = (f(dt))^{\frac{1}{d}}.$$

As we shall see shortly, this transformation has particular stability properties with regard to the differential expressions of our title and this will explain many phenomena on the mechanical and geometrical properties of special curves.

We now return to our central equation

$$f(\theta) + f''(\theta) = cf(\theta)^\alpha$$

, which corresponds to a force law with $\beta = -2 - \alpha$. Owing to its significance and also the importance of the constants α and c , we denote this equation by $\text{ns}(c, \alpha)$.

4.2 The general situation

The starting point of the ensuing discussion was a fact which struck us on reading Newton's Principia, in the version of Chandrasekhar [Ch]. One of the fascinating results he obtains is that when the orbits are conic sections, but with the Sun at the *centre*, then this also satisfies a power law, albeit with $\beta = 1$. These orbits are the images of the Kepler ones under the mapping $z \mapsto z^2$ (cf. Arnold [Ar]). This so-called duality, which apparently was first studied systematically by Kasner although we have been unable to find a precise reference, has been investigated in detail by many authors, see, for example, the same reference. The two families above suggest that there is a certain stability for power laws under transformations of the form $z \mapsto z^n$ and it is this phenomenon which we now investigate. Simple computations show that this is not the case in general, but that for certain special situations, including the ones mentioned above, unexpected coincidences occur in the formulas and this explains the above facts.

Thus our central question is: suppose that f satisfies $\text{ns}(c, \alpha)$. Does g also satisfy such an equation (with different constants) where $g(\theta) = (f(d\theta))^{1/d}$, i.e., g is the d transform of f ? Our above remarks indicate that there is a result of this sort for the special cases:

- a) $f(\theta) = \cos \theta$ (d arbitrary);
- b) $f(\theta) = 1 + e \cos \theta$ ($d = 2$).

As mentioned above, the expressions $f(\theta) + f''(\theta)$ and $f(\theta)^2 + f'(\theta)^2$ will play an important role in our computations.

In the above special cases their values are:

- a) $f(\theta) + f''(\theta) = 0$, $f(\theta)^2 + f'(\theta)^2 = 1$;
b) $f(\theta) + f''(\theta) = 1$, $f(\theta)^2 + f'(\theta)^2 = 2f(\theta) + (e^2 - 1)$.

Now there is a relationship between these two expressions. Indeed

$$\frac{d}{d\theta}(f(\theta)^2 + f'(\theta)^2) = 2f'(\theta)(f(\theta) + f''(\theta)).$$

(This will be discussed in more detail below.)

It follows that if f is a solution to $\text{ns}(c, \alpha)$, then

$$f(\theta)^2 + f'(\theta)^2 = \frac{2cf(\theta)^{\alpha+1}}{\alpha+1} + b$$

for some constant b (or $f(\theta)^2 + f'(\theta)^2 = 2c \ln f(\theta) + b$ if $\alpha = -1$). We emphasise that b depends on the particular orbit that we are examining, within a universe with power law $K \propto \frac{1}{r^{2+\alpha}}$ and “gravitational constant” c . Those orbits with $b = 0$ will be of particular interest below. For the case of a $\frac{1}{r^2}$ law, these are precisely the parabolic orbits.

A simple computation shows that if $g(\theta) = f(d\theta)^{1/d}$ (and so $f(d\theta) = g(\theta)^d$), then, whenever f is a solution of $\text{ns}(c, \alpha)$, we have

$$g^2(\theta) + g'^2(\theta) = \frac{2c}{\alpha+1} g^{2+\alpha d-d}(\theta) + b g(\theta)^{2-2d}$$

for the constant b above and so

$$g(\theta) + g''(\theta) = \frac{c(2 + \alpha d - d)}{\alpha + 1} g(\theta)^{1+\alpha d-d} + b(1 - d)g(\theta)^{1-2d}$$

for $\alpha \neq -1$. (The case $\alpha = -1$ must be dealt with separately).

From these equations we can garner a wealth of information. Firstly we see that the dual of a power law is always the sum of two powers (again for $\alpha \neq -1$).

A particularly interesting case is when $\alpha = 1$ and $n = \frac{1}{2}$. Then we see that a $\frac{1}{r}$ law is dual to one of the form

$$\frac{c_1}{r^2} + \frac{c_2}{r^3}$$

as discussed above.

There are three special situations where the dual collapses to a power law.

- a) $c = 0$, i.e., the case of straight line motion. Then g is a solution of $\text{ns}((1-n)b, 1-2n)$ for any n . This explains the case a) above where $f(\theta) = \cos \theta$ and the dual curves are the sinusoidal spirals.

b) $b = 0$. Then g is a solution of

$$\text{ns} \left(\frac{c(2 + \alpha n - n)}{\alpha + 1}, 1 + \alpha n - n \right)$$

for any n .

This means that $z \mapsto z^d$ transforms a $\frac{2}{d} - 3$ power law into a $2d - 3$ law. This can be used to investigate the duality phenomenon in more detail. Since a great deal has been published on this topic we will not dwell on it but remark that the case where $d = \frac{1}{2}$ corresponds to Newton's result on the duality between conic sections with sun at the centre and such sections with the Sun at a focus, i.e., with $\beta = 1$ and $\beta = -2$ respectively. Another interesting case is where $n = -1$. This maps the case $\beta = -5$ onto itself and thus shows that this case is self-dual (cf., Arnold [Ar]).

As a final remark we note that the computations for the case $\alpha = -1$ are slightly different owing to the presence of the logarithmic term and we leave them to the reader. As a consequence we see that a dual in this case is never a power law (except in the trivial case $d = 1$).

This is significant as it allows us to eliminate the third possibility for dualities.

c) where the two powers $1 + \alpha d - d$ and $1 - 2d$ in the above equation coincide. However, this is the case where $\alpha = -1$ and so no non-trivial duality arises in this manner.

5 A Duality between Trajectories for Central and Parallel Force Laws

As we have seen, the members of the three-parameter family

$$f(t) = p(\cos(d(t - t_0)))^{1/d}$$

are solutions of $\text{ns}(c, \alpha)$ for various values of the parameters c and α , and so the curves of the form $r^d \cos(d\theta) = 1$ all describe planetary orbits which are induced by power laws (we omit the p and t_0 since these correspond to the simple geometrical operations of dilation and rotation respectively).

The main theme of this section is a duality between the curves discussed so far and a second family of curves which have analogous properties with respect to a force parallel to the y -axis, in particular one which is proportional to a power of the distance to the x -axis.

We recall the three motivating examples

1. The cycloid $(t - \cos t, 1 + \sin t)$, which is a brachistochrone and tautochrone for a constant force parallel to the y -axis, or a trajectory for a $\frac{1}{y^2}$ law.
2. The circle $(-\cos t, \sin t)$, which is a geodesic for the Poincaré half-plane.
3. The catenary $y = \cosh x$, which is the form taken on by a hanging chain.

(We remark that each of these curves is a solution of a classical problem of the Calculus of Variations, for which see below. 2. and 3. are also trajectories for $\frac{1}{y^\alpha}$ laws). The connection to the first class of curves is that there is a deep analogy between the properties of curves with parametrisations of the form $(F(t), f(t))$ (where F is a primitive of f) and those of the spirals $rf(\theta) = 1$. Further, both have remarkable properties when f has the special form used in the definition of the MacLaurin spirals. It will be one of the main tasks of this article to display the reasons for these two facts.

We can summarise the above remarks as follows:

- A) The curves of the first family are “spirals” with equations of the form $rf(\theta) = 1$ for suitable functions f of one variable.
- B) The curves of the second family have parametrisations of the form $(F(t), f(t))$ for a suitable function f with F a primitive of f .
- C) Further, when the functions f which occur have the form $f(\theta) = (\cos d\theta)^{\frac{1}{d}}$ for some parameter d (or can be written in this form by simple transformations), then the curves have remarkable properties. As we shall see shortly, this is because these functions are the solutions of three particular ordinary differential equations involving the differential expressions of our title, a fact of some consequence for the mechanical properties of the corresponding curves.

5.1 Curves of the form $(F(t), f(t))$

We therefore consider in more detail parametrised curves of the form

$$(x(t), y(t)) = (F(t), f(t))$$

where F is a primitive of f .

In a certain sense, every generic plane curve with parametrisation $(x(s), y(s))$ can be reparametrised in the form $\left(\int^t f, f\right)$. We simply set $t = \int^s \frac{x'(u)}{y(u)} du$ which means that $\frac{dt}{ds} = \frac{x'(s)}{y(s)}$.

Then if $f(t) = y(s)$ and $F(t) = x(s)$ we have

$$\begin{aligned} \frac{dF}{dt} &= \frac{dF}{ds} \cdot \frac{ds}{dt} = x'(s) \div (x'(s)/y(s)) \\ &= y(s) = f(t). \end{aligned}$$

In particular, if our curve is the graph $(s, y(s))$ of a function, then we have $t = \int^s \frac{1}{y(u)} du$.

We refer to this representation (which is essentially unique) as the *canonical parametrisation* of the curve.

Note that there is no difficulty if we confine attention to curves which lie in the open upper or lower half-plane and whose velocity in the direction of the x -axis never vanishes. If the curve touches or crosses the x -axis or violates the second condition, then care is required and such a parametrisation may not exist. (Think of lines parallel to the y -axis).

We illustrate this with a very simple example: the parabola (s, s^2) . This has canonical parametrisation $(-\frac{1}{t}, \frac{1}{t^2})$. Further examples will be computed below.

Note that the parabola, which is originally “in one piece”, now splits into two parts (corresponding to $t > 0$ and $t < 0$ respectively). This mirrors the fact that the parabola touches the x -axis at its vertex.

5.2 The duality (quadrality)

These considerations make the following concept of duality between curves of the above two families natural:

- A) From spirals to parametrised curves: The spiral $rf(\theta) = 1$ corresponds to the parametrised curve $\left(\int^t f(u) du, f(t)\right)$.
- B) From parametrised curve to spiral: If we are given the curve $(x(s), y(s))$ we compute the canonical parametrisation $(F(t), f(t))$ (i.e., $f(t) = y(s(t))$ with $t = \int \frac{x'(u)}{y(u)} du$) and then associate to it the spiral $rf(\theta) = 1$.

Of course, the transition from the spiral to the parametrised curve is immediate, while the reverse transition requires the intermediary step of computing the canonical parametrisation. We shall find that this procedure sometimes leads to rather surprising results.

We remark at this point that it is an abuse of terminology to talk about a duality between curves since one normally does not distinguish between congruent or even similar curves. However, for our duality the position of the curve with respect to the system of coordinate axes is of crucial importance. Thus the dual of a unit circle with centre at the origin is very different from that of one which passes through the origin. This mirrors the fact that the x -axis and the origin respectively have a privileged role with respect to these two classes of curves. Further, the dual of a curve depends on whether we regard it as a parametrised curve or as a “spiral”. Another aspect of the duality which will be of some consequence later is the fact that although the transformation from a curve to its dual does not arise from a transformation of space, nevertheless the transformation *does* act pointwise on the curves themselves. Specifically, the point $(F(t), f(t))$ on the parametrised curve is associated with $\frac{(\cos t, \sin t)}{f(t)}$ on the dual spiral.

For many purposes it will be useful to extend this duality to a quadrality by adding the inverse of $rf(\theta) = 1$ in the unit circle (i.e., the curve $r = f(\theta)$) and its dual, i.e., $\left(\int^t g(u), g(t)\right)$ where $g(t) = \frac{1}{f(t)}$. Thus we have the scheme

$$\begin{array}{ccc} rf(\theta) = 1 & \leftrightarrow & (F(t), f(t)) \\ \updownarrow & & \updownarrow \\ r = f(\theta) & \leftrightarrow & \left(\int \frac{1}{f(u)} du, \frac{1}{f(t)}\right) \end{array} .$$

The main purpose of our note is to explain the significance of this duality, in particular for families which are trajectories in the above generalised sense, one with respect to physical properties with regard to r^α laws, the other with respect to y^α laws.

5.3 Examples of duality

I. We begin with duality for spirals. Note that we are using the term spirals in the rather loose sense of any curve with an equation of the form $rf(\theta) = 1$. Since the computation of the dual of a spiral requires only an integration, it is an easy task to set up a program, say in Mathematica, which automatically computes the inverse and both duals and displays plots of all four curves. We will therefore only mention a few cases which are of particular interest. We

begin with the Archimedes spiral $r = a\theta$, i.e., where $f(\theta) = \frac{1}{a\theta}$. Its inverse is $r = \frac{1}{a\theta}$, that is, $f(\theta) = a\theta$, which is known as the hyperbolic spiral.

The dual curves have parametrisations $\left(\frac{1}{a} \ln t, \frac{1}{at}\right)$ and $\left(\frac{1}{at^2}, at\right)$ respectively, i.e. they are the graph of $y = \frac{1}{a}e^{-ax}$ and the parabola $y^2 = 2ax$.

II. As noted above, the canonical representation for the parabola (s, s^2) is $\left(-\frac{1}{t}, \frac{1}{t^2}\right)$ and so the dual spiral is $r = \theta^2$, with inverse spiral $r\theta^2 = 1$ which dualises to Neil's parabola $\left(\frac{t^3}{3}, t^2\right)$.

III. The logarithmic spiral has the form $rae^{b\theta} = 1$, i.e., has $f(\theta) = ae^{b\theta}$. Its inverse is $\frac{r}{a}e^{-b\theta} = 1$ and so is again a logarithmic spiral.

Their duals are the curves with parametrisations $\left(\frac{a}{b}e^{bt}, ae^{bt}\right)$ and $\left(-\frac{1}{ab}e^{-bt}, \frac{1}{a}e^{-bt}\right)$, i.e., rays of the lines $y = bx$ and $y = -bx$ respectively

IV. The generalised parabola $y = ax^\alpha$. In this case, the reparametrisation is

$$\left(((1-\alpha)t)^{\frac{1}{1-\alpha}}, ((1-\alpha)t)^{\frac{\alpha}{1-\alpha}}\right).$$

When $\alpha < 1$ the parameter t ranges over the positive half-line, and when $\alpha > 1$ over the negative half-line.

This is a special case of the so-called higher-order spirals, which are written classically as $r^k = \frac{\theta}{2\pi}$. (For our purposes, k can be any real number, not necessarily an integer). Its inverse is also a higher order spiral, and the dual curves are higher order parabolas, i.e., of the form $y \propto x^\alpha$ for a suitable α .

Of particular interest is the case $k = 2$, which is known as the Fermat or parabolic spiral.

Further interesting spirals are the Galilei spirals $r = a - b\theta^2$, the logarithmic spirals $r = ae^{b\theta}$ and the rose curves $r = a \sin b\theta$. The special rose curve with $b = 1$ is one of the Cotes' spirals and is known as the epispiral.

It is natural to generalise the rose curves to the family $r^d = a^d \sin(bd\theta)$ which generalises the family of MacLaurin spirals and has some of their properties, albeit in a weaker form (for example, they are orbits for force laws which are the sum of two powers).

5.4 The sinusoidal spirals and catenaries

The specific properties of the sinusoidal spirals

$$r^d \cos(d\theta) = 1$$

suggest that their duals, i.e., the curves with parametrisation

$$(F_d(t), f_d(t))$$

where $f_d(t) = (\cos dt)^{\frac{1}{d}}$ and F_d is a primitive of f_d , will also possess interesting properties. It turns out that this is, in fact, the case, as we shall see shortly. We propose to call these curves *MacLaurin* or *sinusoidal catenaries* since they arise from the classical catenary using a transformation which will be discussed below.

The key to this phenomenon lies in the fact that the functions $f(t) = a(\cos d(t - t_0))^{\frac{1}{d}}$ have special properties—namely they are solutions of differential equations of the form

$$\begin{aligned} f(t)^2 + f'(t)^2 &= af^\alpha \\ f(t) + f''(t) &= bf^\beta \\ f(t)f''(t) - f'(t)^2 &= cf^\gamma \end{aligned}$$

as we shall see below.

The expressions $f^2 + f'^2$, $f + f''$ and $ff'' - f'^2$ occur frequently in computations involving the geometric and mechanical properties of curves (we have already met the first two). Hence those cases where they are proportional to a power of f (and so to a power of the distance to the origin or to the x -axis respectively) can be expected to have special properties.

It is not a coincidence that the above family of functions satisfies each of these equations since the latter are closely related, as we shall see shortly.

We note that the usual rule of thumb prepares us to anticipate ∞^3 or ∞^4 resp. ∞^4 solutions for these equations (one or two constants of integration and the two parameters in the equation). Thus we would expect our list to include all of the solutions in the first case, but not in the other two. In fact, it is also possible to give all of the solutions in the third case explicitly using elementary functions, and we shall do this in the next section.

6 On the expressions $f^2 + f'^2$, $f + f''$, $ff'' - f'^2$, how to solve the equations $f^2 + f'^2 = af^\alpha$, $f + f'' = bf^\beta$, $ff'' - f'^2 = cf^\gamma$ and why one might want to

We are interested in the three differential equations

$$(A_1) \quad f^2 + f'^2 = af^\alpha;$$

$$(B_1) \quad f + f'' = bf^\beta;$$

$$(C_1) \quad ff'' - f'^2 = cf^\gamma.$$

It will be convenient to consider the following more general situation:

$$(A) \quad f^2 + f'^2 = \phi(f)$$

$$(B) \quad f + f'' = \psi(f)$$

$$(C) \quad ff'' - f'^2 = \rho(f)$$

where ϕ , ψ and ρ are given smooth functions and the unknown f is a function of one variable (so that our original equations correspond to the situation where the functions ϕ , ψ and ρ are powers of f (with a constant of proportionality)).

The reason for this interest and the relevance of these equations for geometric and mechanical properties of our special types of curves will be discussed below.

The above stability property is a consequence of a simple computation which shows that if g is the d -transform of f , i.e., $g(t) = f(dt)^{1/d}$, then

$$\begin{aligned} g^2 + g'^2 &= f^{\frac{2}{d}-2}(f^2 + f'^2) \\ g + g'' &= f^{\frac{1}{d}-2}((f^2 + f'^2) + d(ff'' - f'^2)) \\ gg'' - g'^2 &= f^{\frac{2}{d}-2}d(ff'' - f'^2). \end{aligned}$$

Hence if f is a solution of (A), (B) or (C), then g is a solution of the corresponding equations with the expressions $g^{2-2d}\phi(g^d)$, or $g^{1-2d}(\phi(g^d)g^d + d\rho(g^d))$ and $dg^{2-2d}\rho(g^d)$ respectively as the right-hand sides.

6.1 The relationship between the equations

These equations and their solutions are closely related, as we shall now see.

1) If f is a solution of (A), i.e., if $f^2 + f'^2 = \phi(f)$, then, differentiating and simplifying, we get

$$2ff' + 2f'f'' = \phi'(f)f'$$

and so $f + f'' = \frac{1}{2}\phi'(f)$, i.e., f is a solution of (B) with $\psi(f) = \frac{1}{2}\phi'(f)$. Also

$1 + \frac{f'^2}{f^2} = \frac{\phi(f)}{f^2}$, and so, again by differentiating and simplifying, we get

$$(ff'' - f'^2) = \frac{1}{2}f\phi'(f) - \phi(f),$$

i.e., f is a solution of (C) with $\rho(f) = \frac{1}{2}f\phi'(f) - \phi(f)$.

Now assume this f is a solution of (B), i.e., $f + f'' = \psi(f)$. Then by the above $f^2 + f'^2 = 2\phi(f)$, where ϕ is a primitive of ψ . Hence

$$ff'' - f'^2 = \frac{1}{2}f\psi(f) - \phi(f),$$

an equation of the form (C). Finally, if f is a solution of (C) then we can solve the (linear) differential equation

$$\frac{1}{2}f\phi'(f) - \phi(f) = \rho(f)$$

for ϕ , given ρ , and so obtain an equation of type (A) for f .

6.2 Some solutions

In fact, using standard elementary techniques for solving O.D.E.s by quadrature, we can express the solutions implicitly as follows:

- | | | |
|-------------------|------------------------------|---|
| (A) | $f^2 + f'^2 = \phi(f)$ | $f(t) = F^{-1}(t + c)$, where F is a primitive of $(\phi(u) - u^2)^{-\frac{1}{2}}$ |
| (A ₁) | $f^2 + f'^2 = af^\alpha$ | special case of (A) |
| (A ₂) | $f^2 + f'^2 = af^\alpha + d$ | special case of (A) |
| (B) | $f + f'' = \psi(f)$ | special case of (A) with $\phi = 2 \int \psi$ |
| (B ₁) | $f + f'' = bf^\beta$ | special case of (B) |
| (C) | $ff'' - f'^2 = \rho(f)$ | $f(t) = \exp(G^{-1}(t + c))$, i.e., $\ln f(t) = G^{-1}(t + c)$
where G is a primitive of $(2 \int \rho(e^u)e^{-2u})^{-\frac{1}{2}}$ |
| (C ₁) | $ff'' - f'^2 = cf^\gamma$ | special case of (C). |

We repeat for the record that if $f(t) = p(\cos d(t - t_0))^{\frac{1}{d}}$ then

$$\begin{aligned} f^2 + f'^2 &= p^2 f^{-2d+2} \\ f + f'' &= -p^2(d-1)f^{-2d+1} \\ f f'' - f'^2 &= -p^2 d f^{-2d+2}. \end{aligned}$$

Thus this f solves (A₁) with $a = p^2$, $\alpha = -2d + 2$
 (B₁) with $b = -p^2$, $\beta = -2d + 1$
 and (C₁) with $c = -p^2$, $\gamma = -2d + 2$.

It is interesting that the hyperbolic trigonometric functions have similar but not identical properties, namely if $g(t) = p(\cosh d(t - t_0))^{\frac{1}{d}}$, then

$$\begin{aligned} g^2 - g'^2 &= p^2 g^{-2d+2} \\ g - g'' &= -p^2(d-1)g^{-2d+1} \\ g g'' - g'^2 &= p^2 d g^{-2d+2}. \end{aligned}$$

We will mention a possible application of these facts below.

The noteworthy form of the above solutions shows that they all arise from the simple function $\cos t$ by applying the d -transformation $f(t) \rightsquigarrow f(dt)^{\frac{1}{d}}$ which, we recall, corresponds to the transformation $z \mapsto z^d$ of the complex plane. This fact and, in particular, the fact that the latter is conformal, is relevant for some of the remarkable properties of the families (see, e.g. [Sc2] where this is used to explain many properties of the MacLaurin spirals).

The reason for the above stability property lies in a simple computation which shows that if $g(t) = f(dt)^{1/d}$, then

$$\begin{aligned} g^2 + g'^2 &= f^{\frac{2}{d}-2}(f^2 + f'^2) \\ g + g'' &= f^{\frac{1}{d}-2}((f^2 + f'^2) + d(f f'' - f'^2)) \\ g g'' - g'^2 &= f^{\frac{2}{d}-2}d(f f'' - f'^2). \end{aligned}$$

Hence if f is a solution of (A), (B) or (C), then g is a solution of $g^2 + g'^2 = g^{2-2d}\phi(g^d)$, or $g + g'' = g^{1-2d}(\phi(g^d) + d\rho(g^d))$, or $g g'' - g'^2 = d g^{2-2d}\rho(g^d)$.

In order to motivate these considerations we shall now show that the above three differential expressions are ubiquitous in calculations which are related to mechanical or geometric properties of curves.

6.3 Where the expressions arise

As mentioned above we have two central motivations for these considerations. Firstly, the expressions of our title occur in many formulae which arise in analytically describing geometrical or mechanical properties of curves described as above. Secondly, and as a consequence, for curves which correspond to functions with the property that they are solutions of the above equations, then these quantities will take on a particularly simple form and so the curves will have remarkable geometric and mechanical properties. We believe that this is the natural explanation of the special properties of the MacLaurin spirals and catenaries.

We bring a sample of the kind of occurrence that we have in mind:
For the spiral $rf(\theta) = 1$, then

1. the curvature is $\frac{f^3(f + f'')}{(f^2 + f'^2)^{\frac{3}{2}}}$;
2. the affine curvature is $f^3(f + f'')$ (see below);
3. if a planet moves around the curve in such a manner that it obeys Kepler's second law, then the absolute value of the acceleration vector is $f^2(f + f'')$;
4. the curvature of the hodograph is $(f + f'')^{-1}$;
5. the infinitesimal length is $ds^2 = \frac{1}{f^4}(f^2 + f'^2)d\theta^2$.

For the curve with parametrisation $(F(t), f(t))$,

1. the curvature is $\frac{ff'' - f'^2}{(f^2 + f'^2)^{3/2}}$;
2. if it moves under a force parallel to the y -axis which depends only on the distance to the x -axis, then the absolute value of its acceleration is $\frac{ff'' - f'^2}{f^3}$;
3. The infinitesimal length is given by the formula $ds^2 = (f^2 + f'^2)dt^2$.

A further situation where the expression $f^2 + f'^2$ occurs will now be discussed briefly.

6.4 The calculus of variations

One of the most important aspects of the sinusoidal spirals and catenaries is that they provide a plethora of solutions to natural problems of the variational calculus. Elementary treatises on this subject typically use the following model applications, which are mathematically interesting, are of great historical interest and can be computed explicitly: the catenary (the curve which minimises potential energy and whose surface of revolution has a minimal-area property), the cycloid (brachistochrone), Dido circles (curves of a given length with endpoints on the x -axis which enclose maximal area). We shall now show that these are special cases of a general phenomenon and that this explains many of the remarkable properties of the MacLaurin spirals and catenaries.

The common structure of these problems is that they maximise or minimise functionals of the form

$$\int y^\delta ds \quad \text{or} \quad \int r^\gamma ds.$$

We consider the second case: If we use our basic parametrisation

$$\frac{(\cos(t), \sin(t))}{f(t)}$$

then it reduces to the variation problem of minimising or maximising a functional of the form

$$\int F_\beta(x, f(x), f'(x)) dx$$

where $F_\beta(x, u, v) = u^\beta(u^2 + v^2)^{\frac{1}{2}}$ for a suitable β ($\beta = \delta$ resp. $\beta = -\gamma - 2$). We now use the simple fact that for each β , f is a solution of the corresponding Euler-Lagrange equation if and only if $f^2 + f'^2 = af^\alpha$ for some a where $\beta + 2 = \frac{\alpha}{2}$. This is seen by using the standard fact that in the case, as here, where the corresponding integrand is independent of the first variable, the Euler equation has the form: $vf_3 - f$ is constant. This expression is

$$\frac{u^{\beta+2}}{a^{\frac{1}{2}} u^{\frac{\alpha}{2}}}$$

when $f^2 + f'^2 = af^\alpha$, and so is constant when the above relationship holds.

This explains many of the remarkable mechanical properties of the MacLaurin spirals.

For example, in [Br] it is shown that the ∞^2 family of parabolae of the form

$$\alpha(y - \alpha) = (x - d)^2$$

are the paths of light rays in a medium with refraction index $\propto y^{\frac{1}{2}}$. This is a special case of the above result since the canonical parametrisation of the above curve is

$$\left(2\alpha \tan \frac{t}{2} + d, \alpha \left(\cos \frac{t}{2} \right)^{-2} \right)$$

and for $f(t) = (\cos \frac{t}{2})^{1/2}$, we have $f^2 + f'^2 = f^3$.

We now show that the MacLaurin catenaries are also solutions of suitable variational problems. Interestingly, although the concrete variational problems which are usually featured as text book examples are usually formulated for the case of a parallel force, many more concrete examples are known for the case of a central one. However, using the methods expounded here, we can extend the list of such examples for parallel forces considerably.

We consider the problem of maximising

$$\int y^\delta ds$$

i.e. the variational problem with kernel

$$J(y, y') = y^r (1 + y'^2)^{-\frac{1}{2}}.$$

The corresponding Euler equation reduces to

$$y^r (1 + y'^2)^{-\frac{1}{2}} = c$$

where c is an arbitrary constant.

If we now use the parametrisation $(F(t), f(t))$, this reduces to $f^{r+1}(f^2 + f'^2)^{-\frac{1}{2}} = c$.

Hence if $f^2 + f'^2$ is proportional to f^α , then the above parametrisation provides a solution whenever $r + 1 = \frac{\alpha}{2}$. Hence we obtain a solution with $f(t) = p(\cos dt)^{\frac{1}{d}}$ when $2 - 2d = \alpha$.

6.5 Two more subtle examples

As we have seen above, Dido circles and cycloids are examples of MacLaurin's catenaries. But why are catenaries, i.e., the graphs of the functions $a \cosh\left(\frac{x}{a}\right)$ also members of this family, as the name implies? This is because they have canonical parametrisations $(F_{-1}(t), f_{-1}(t))$ as the reader can verify.

A further fascinating class of curves are formed by the two parameter family of parabolae of the special form $y = a + \frac{(x-d)^2}{4a}$, which are the paths taken by light rays in suitable media. Once again, these are MacLaurin catenaries, explicitly $(F_{-2}(t), f_{-2}(t))$. This explains the following remarkable fact from [2]: The above family of parabolae are the paths followed by light rays in a medium where the index of refraction $n(x, y)$ is proportional to $y^{-\frac{1}{2}}$. Our method allows us to solve the corresponding problem for $n(x, y) \propto y^\alpha$ for any α .

6.6 Miscellanea

In this section we bring some miscellaneous remarks on our main topic.

6.6.1 The explicit form of the parametrisation $(F_d(t), f_d(t))$

The reader will have noticed that we never required the explicit form of the primitive of f_d and so the explicit parametrisation of the MacLaurin catenaries. For the record we include it here:

$$x(t) = -\frac{1}{1+d}(\cos(dt))^{1+\frac{1}{d}}\operatorname{cosec}(dt)F_1^2\left(\frac{1}{2}\left(1+\frac{1}{d}\right), \frac{1}{2}, \frac{1}{2}\left(3+\frac{1}{d}\right), (\cos(dt))^2\right)$$

$$y(t) = (\cos^d(dt))^{\frac{1}{d}}$$

(we used Mathematica to obtain this expression— F_1^2 is the hypergeometric function.)

6.6.2 The connection with contact geometry

The representation of a curve in the form $(F(t), f(t))$, which might seem rather artificial and *ad hoc*, is natural within the framework of contact geometry as was pointed out to me by Valentin Lychagin. Namely, if we consider the simplest contact manifold $\mathbf{R}^3 = \{(x, y, t) : x, y, t \in \mathbf{R}\}$ with the natural structure given by the differential form $\omega = dx - y dt$, then the Legendrian submanifolds are the curves of the form $((F(t), f(t), t))$ i.e. our special parametrisation displays the curve as the projection of such a manifold onto the (x, y) coordinates.

6.6.3 Curves with prescribed curvature

Many problems in applied mathematics can be subsumed under the general one: given a function $f(x, y)$ on the plane or a subset thereof, we are required to find a parametrised curve such that at each point (x, y) on the curve the curvature κ there is given by $\kappa = f(x, y)$. In other words we are looking for

solutions of the equation

$$\frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}} = f(x(t), y(t)).$$

It follows from above that the sinusoidal spirals are solutions of this problem for the case $\kappa \propto r^\alpha$ for corresponding α . This fact, which was known to the classical differential geometers (cf. [Go], [Lo], has been rediscovered (for special cases) several times (cf. e.g. [Si]). Thus MacLaurin had shown that the radius of curvature of the spiral is given by the formula

$$R = \frac{a^n}{(n+1)r^{n-1}}.$$

Special cases of these results can be found in the modern literature. All these results follow easily from the theory developed here. Analogously, the Maclaurin catenaries are solutions of related problems for $\frac{1}{y^\alpha}$ laws.

All these results follow easily from the theory developed here. Analogously, the Maclaurin catenaries are solutions of related problems for $\frac{1}{y^\alpha}$ laws, as we shall show below.

6.6.4 The rectification problem

If $(x(t), y(t)) = \left(\frac{\cos t \sin t}{f(t)}, \frac{1}{f(t)}\right)$ is the canonical parametrisation of the curve $rf(\theta) = 1$ then $ds^2 = \frac{1}{f^4}(f^2 + f'^2) dt^2$. If $(x(t), y(t)) = (F(t), f(t))$ then $ds^2 = (f^2 + f'^2) dt^2$. For $\left(\int^t \frac{1}{f(u)} du, \frac{1}{f(t)}\right)$ we have $ds^2 = \frac{1}{f^4}(f^2 + f'^2) dt^2$. Hence the expression for the infinitesimal length of the spiral $rf(\theta) = 1$ and for its “diagonal dual” $(G(t), g(t))$, where $g = \frac{1}{f}$ and G is a primitive of g , coincide. This implies that the length between two points on the spiral and the corresponding ones on the diagonal dual are the same. This is the background for many rectification results in the classical theory. For example, if $r = a\theta$, $f(\theta) = \frac{1}{a\theta}$ and so the dual curve is $\left(\frac{1}{a} \ln t, \frac{1}{at}\right)$, i.e., the logarithmic spiral $y = \frac{1}{a}e^{-ax}$. The polar reciprocal is $ra\theta = 1$ with dual $\left(\frac{at^2}{2}, at\right)$, i.e., the parabola $y^2 = 2ax$. In the words of [Lo] this means that the rectification problem for the Archimedean spiral is equivalent to the analogous problem for the parabola. The rectification problem played an important role in the

history of geometry, for which see the same reference. Since this question seems rather innocuous from a modern point of view, it is perhaps of interest to recall the historical perspective as described there:

Thus we see that Archimedes had solved two of the three fundamental questions for his spirals, namely the determination of its tangent and of the area swept out by a radial vector moving along the curve, but did not touch on the rectification problem. This important task could not be solved by his contemporaries or immediate successors, and would only be successfully attacked nearly two thousand years later, independently by Cavalieri, St. Vincentius, Roberval, Pascal and Fermat” (our translation).

The method used here immediately give a plethora of such results. We mention only one such, due to Fermat: the rectification problems for the spiral $r^k = \frac{a^k \theta}{2\pi}$ and the generalised parabola $z^{k+1} = (k+1)p^k x$, when $a^k = 2k\pi p^k$ ([10], p. 435), are equivalent.

6.6.5 The Weingarten mapping of surfaces of revolution

We consider the surface of revolution

$$\phi(u, v) = (f(v) \cos u, F(v), f(v) \sin u)$$

obtained by rotating the curve $(0, F(v), f(v))$ around the x -axis. Then if we compute the coefficients of the two fundamental forms, we find that

$$E = f^2, \quad F = 0, \quad G = f'^2 + f^2$$

and

$$L = f^2/(f^2 + f'^2)^{1/2}, \quad M = 0, \quad N = (f'^2 - ff'')/(f^2 + f'^2)^{1/2}.$$

Once again, we see the prevalence of the three expressions

$$f^2 + f'^2, \quad f + f'' \quad \text{and} \quad ff'' - f'^2.$$

As a consequence, if we rotate the MacLaurin catenary $(F_d(t), f_d(t))$ around the x -axis, then all of the quantities involved, i.e., E , G , L and N (and, as a consequence, the principal curvature and Gaußian curvature), are proportional to powers of the distance from the x -axis. (F and M vanish, of course).

Without going into details we note that the Christoffel symbols for these surfaces (and hence the geodetic equations) also take on a very simple form.

If instead of using a classical Euclidean rotation we employ a hyperbolic one, i.e. the linear mapping with matrix

$$\begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix},$$

we obtain corresponding results for the Minkowski pseudo-metric (i.e., the one corresponding to the quadratic form $x^2 + y^2 - z^2$), which suggests that such surfaces could be of interest in the theory of relativity (this follows from the relationships satisfied by the hyperbolic trigonometric functions displayed above).

6.6.6 Remark

The characterisation of the functions $p(\cos(d(t - t_0)))^{1/d}$ as solutions of an equation of the form $f^2 + f'^2 = af^\alpha$ is implicit in MacLaurin's treatment. He defines the spirals as curves which satisfy the condition

$$\tan V = r \frac{d\theta}{dr},$$

where V is the angle between the vector from the origin to the point (r, θ) on the curve and the tangent there. This condition is equivalent to the differential equation

$$\frac{d\theta}{dr} = \frac{r^{n-1}}{\sqrt{a^{2n} - r^{2n}}}$$

(cf. Gomes Teixeira [Go], p. 259, but note that there is a misprint in the equation there). If we put $f(\theta) = \frac{1}{r(\theta)}$, then the above equation reduces to

$$f^2 + f'^2 = a^{2n} f^{2n-2}.$$

7 The trajectories for parallel power laws

We now turn to a special case of the following problem: given a force field (in our examples on a subset of the plane), determine the family of all trajectories of particles moving under this field. An analogous problem is: given a function on a subset of the plane, determine the two-parameter family of curves which are such that at each point on the curve, the curvature corresponds to the value of this function. (Such problems arise, e.g., in theory of elasticae—cf. [Si]). We will give explicit formulae for the genuinely planar trajectories of particles which move under a force law of the form $K \propto y^\alpha$.

In a second section we investigate free fall along a line parallel to the y -axis for such laws. In the final section we show that analogous methods can be used to solve the curvature problem, again for the case where the curvature is proportional to a power of the distance from the x -axis. We are using the term “explicit” in the following precise sense. In the first case, we show that the trajectories can be parametrised by functions which are called elementary in normal usage (we include the hypergeometric functions in this category). In the two second cases, we have to allow the use of the inverses of such functions.

More precisely, we shall show that the trajectories for a y^α -law are the curves with parametrisations $(F_d(t), f_d(t))$, where f_d is a function of one of four particular forms for suitable parameters a , b and c , F_d is a primitive of f_d and d is $\alpha + 1$, where α is the index of the power law.

The four forms of f_d are

$$f_d(t) = \left(\frac{1}{ad^2t^2} \right)^{\frac{1}{d}} ;$$

for fixed d , this generates a two-parameter or ∞^2 family of curves (the parameters are a and an integration constant). The parameter t ranges over the punctured line, i.e. $\mathbf{R} \setminus \{0\}$. (f_d is, of course, proportional to $t^{-\frac{2}{d}}$, but we leave it in the above form for reasons which will soon become obvious).

$$f_d(t) = \left(\frac{a(b-c)^2 e^{a(b+c)dt}}{(e^{abdt} + e^{acdt})^2} \right)^{\frac{1}{d}} ,$$

a three-parameter family. Here t ranges over the whole real line.

$$f_d(t) = \left(\frac{a(b-c)^2 e^{a(b+c)dt}}{(e^{abt} - e^{acdt})^2} \right)^{\frac{1}{d}} ,$$

a three-parameter family. Once again, t ranges over the punctured line.

$$f_d(t) = (ac^2 \sec^2(acdt)^2)^{\frac{1}{d}} ,$$

a three-parameter family (a , c and an integration constant). t ranges over the set of values for which $acdt$ lies in the open interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

The reader will have noticed that there is a problem with a $\frac{1}{y}$ law, i.e. where $d = 0$ and we discuss this case later.

These are parametrisations (in the differential geometric sense) of the curves traced out by a particle moving under the corresponding force law—they do not describe its actual motion, i.e., t is not time. The parametrised motion is $(u, f \circ F^{-1}(u))$, and cannot, in general, be written out explicitly using elementary functions but also requires the inverse of such a function.

We think that it is of some interest to document the fact that we can write down explicitly *all* the trajectories, using simple elementary functions (and an integration), in the case of a parallel power law. We remark that these integrations can be carried out by Mathematica and the results can also be expressed in terms of elementary functions (if one admits this status to the hypergeometric functions). We have added the explicit forms at the end of the section. In contrast to the case of parallel laws, there are, for central power laws, just three cases in which all of the trajectories can be described explicitly using elementary functions: the Kepler case $K \propto r^{-2}$ mentioned above, Hooke's law $K \propto r$ (where the orbits are conic sections with centre at the origin) and the so-called Cotes' spirals ($K \propto r^{-3}$). In general, the trajectories for a given force law form a three-parameter or ∞^3 family. For the general central power law MacLaurin produced an explicit ∞^2 family of trajectories (later baptised as the MacLaurin spirals—cf. [Co1]), but the remaining ones can only be described indirectly in the general case as far as we know (using functions which can be determined implicitly after a quadrature). Thus in the Kepler case, MacLaurin's family only picks up the parabolic orbits.

The special case of rectilinear motion turns out, perhaps surprisingly, to be more intricate, and it is interesting to note that Newton, in his Principia, devoted a whole section to this case, which he regarded as a limiting case of the planar one (for a central force). In this case, it is, of course, not the geometrical form of the motion which is of interest, but its direct description, i.e., formulae for the position as a function of time. Here the results are less satisfactory in the sense that we have to use not just elementary functions but also the inverse of such a function.

For a brief introduction to the curvature problem and its relationship to the Euler elasticae, see Singer [Si]. A similar proviso applies to our solutions here.

7.1 Planar motion under a parallel law

We will now briefly discuss the method which leads to the above formulae. Once one has them, then it is a routine application of the chain rule to show that the curves they describe actually have the claimed property, i.e., that they are trajectories for suitable laws. The only interesting fact is to show

how these formulae arise, and this we shall now do. Since we are considering y^α laws, then it is natural to confine attention to curves in the upper half-plane. In the Galilean situation, we must consider four cases, depending on the position of the parabola with respect to the x -axis, exemplified by the graphs of the following functions:

$$y = x^2$$

(a full parabola touching the x -axis);

$$y = x(1 - x)$$

with x between 0 and 1 (an arc of a parabola with vertex pointing upward);

$$y = x(x - 1)$$

with two unbounded branches corresponding to $x < 0$ and $x > 1$;

$$y = x^2 + 1$$

(a parabola which is disjoint from the x -axis).

Of course, only 6) is physical, i.e., corresponds to an attractive force towards the x -axis. The other three describe trajectories under a repulsive force (anti-gravity).

We remark firstly that in this section we are excluding the cases of free fall parallel to the y -axis, which implies that we can parametrise the trajectories in the form $(F(t), f(t))$, where F is a primitive of f , as shown above. Our proof is a combination of the following simple observations:

- A) If we parametrise a curve in the special form $(F(t), f(t))$ as above (i.e. where F is a primitive of f) and a particle moves along this curve under a parallel force law, then the acceleration at the point $(F(t), f(t))$ is $\left(0, \frac{f(t)f''(t) - f'(t)^2}{f^3(t)}\right)$.

This is a simple consequence of the chain rule.

Hence

- B) We have a y^α law if and only if f satisfies an equation of the form $f(t)f''(t) - f'(t)^2 = bf(t)^\beta$ for constants b and β . The relationship between this β and the exponent α of the power law is very simple— $\beta = \alpha + 3$. Of course, the force is attractive or repulsive (with reference to the x -axis) according to whether b is negative or positive.
- C) If f satisfies an equation as in B), then so does its d -transform f_d (with distinct parameters b and β). This follows from the equation:

$$f_d f_d'' - f_d'^2 = d f^{-2+\frac{2}{d}} (f f'' - f'^2)$$

which we gave above.

The crucial point is now the following rather unorthodox parametrisations of the parabola:

- D) The general parabola (more precisely, the part or parts above the x -axis) has canonical parametrisation $(F(t), f(t))$ where f has one of following four forms:

$$f(t) = \frac{1}{at^2};$$

$(F(t), f(t))$ is then a parametrisation of the parabola $y = a(s - b)^2$ where we choose F so that $F(0) = b$. (Again, t ranges over the punctured reals and the vertex is “lost”);

or

$$f(t) = \frac{a(b - c)^2 e^{a(b+c)t}}{(e^{abt} + e^{act})^2};$$

$(F(t), f(t))$ is then a parametrisation of the parabola $y = a(x - b)(c - x)$, or more precisely of the arc in the upper half plane. t ranges over the real line;

or

$$f(t) = \frac{a(b - c)^2 e^{a(b+c)t}}{(e^{abt} - e^{act})^2};$$

$(F(t), f(t))$ then parametrises $y = a(x - b)(x - c)$, or more precisely the two unbounded branches in the upper half plane. They correspond to $t < 0$ and $t > 0$ respectively;

or

$$f(t) = ac^2 \sec^2(at)$$

where $(F(t), f(t))$ parametrises $y = a((x - b)^2 + c^2)$. t ranges over the open interval $] -\frac{\pi}{2a}, \frac{\pi}{2a}[$.

(The four different forms correspond to the four possible positions of the parabola with regard to the x -axis, as described above).

In these parametrisations, a is assumed to be positive and $b < c$.

E) in each of the cases in D) the function f satisfies the condition

$$f(t)f''(t) - f'(t)^2 = 2af(t)^3.$$

We have thus verified Galilei's result that the parabolas are orbits under a constant parallel force law, albeit in a rather roundabout manner. However, the payoff from this approach now follows.

F) If we combine this fact and the formula in C) then we see that the functions quoted at the beginning of the article (which arise from the above parametrisations of the parabola via the process considered there) satisfy the condition that $ff'' - f'^2$ is proportional to a power of f , in fact to f^{2+d} , and so the curves mentioned in the first paragraph are trajectories for $K \propto y^{d-1}$. Of course, as mentioned above, one can compute this directly; but the treatment described here shows how to find the appropriate form for the functions f .

We now obtain the results above as follows. We suppose that we have a trajectory for a y^α law parametrised in the form $(G(t), g(t))$. Then for any α we can find a suitable d so that $g(t) = f_d(t) = f(dt)^{1/d}$, where f corresponds to the Galilean case, i.e., a constant force. We then use the above formula for the possible forms for f and so find g as above.

It thus only remains to consider the singular case, that of a $\frac{1}{y}$ law. In this case, the differential equation for f (i.e. $ff'' - f'^2 = cf^2$) can be solved directly, and one sees that the trajectories are $(F(t), f(t))$, where f is a function of the form $\exp p(t)$ with p a quadratic function.

For the sake of completeness, we add the explicit forms of the functions F_d , i.e., the primitives of the corresponding f_d for cases 2), 3) and 4) at the

head of the article (the case 1) is trivial):

$$\begin{aligned}
F_d(t) &= \left(\frac{e^{dt}}{(1+e^{dt})^2} \right)^{\frac{1}{d}} (1+e^{dt})^{2/d} {}_2F_1 \left(\frac{1}{d}, \frac{2}{d}; 1 + \frac{1}{d}; -e^{dt} \right) \\
F_d(t) &= \frac{1}{a(b-c)} \left((1 - e^{a(b-c)dt})^{2/d} \left(\frac{a(b-c)^2 e^{a(b+c)dt}}{(e^{abdt} - e^{acdt})^2} \right)^{\frac{1}{d}} \right. \\
&\quad \left. {}_2F_1 \left(\frac{1}{d}, \frac{2}{d}; 1 + \frac{1}{d}; e^{a(b-c)dt} \right) \right) \\
F_d(t) &= \frac{1}{2ad} \left(\cos^2(ad t)^{\frac{1}{d}-\frac{1}{2}} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{d}; \frac{3}{2}; \sin^2(ad t) \right) \right. \\
&\quad \left. (ac^2 \sec^2(ad t))^{\frac{1}{d}} \sin(2ad t) \right).
\end{aligned}$$

These expressions were computed with Mathematica (in the first equation we have assumed that $a = 1$, $b = 0$ and $c = 1$ to make the formula more tractable).

In the singular case (i.e. a $\frac{1}{y}$ law), the formula for $F(t)$ requires the use of the error function from statistics. Since the explicit form depends on the nature of the quadratic function, i.e., on whether it is definite or indefinite, we omit the details.

As final remarks, we note firstly that despite the complex form of these formulae, they do, of course, specialise to simple curves for certain choices of the parameters. For example a Dido circle (that is, a circle with centre on the x -axis) is a trajectory for a $\frac{1}{y^3}$ law (interestingly, this case was treated by Newton in his Principia). Secondly, using the theory developed here, it is easy to derive a criterion for a given curve with parametrisation $(x(s), y(s))$ to be the trajectory of a power law. It is that $\frac{A(s)}{B(s)}$ be constant, where

$$\begin{aligned}
A(s) &= (3y'(s)y''(s) + y(s)y'''(s))x'(s)^2 - (3x''(s)y'(s)^2 + y(s)x'''(s)y'(s) \\
&\quad + 3y(s)x''(s)y''(s))x'(s) + y(s)y'(s)x''(s)^2 \\
B(s) &= x'(s)y'(s)(x'(s)y''(s) - x''(s)y'(s)).
\end{aligned}$$

The index of the power law is then this constant minus three.

Using this criterion one can check that not only are Dido circles trajectories for the y^{-3} power law, but also that, as mentioned above, they are the only circles which are trajectories for *any* power law. Ellipses and hyperbolae (not necessarily rectangular) with the x -axis as axis are also trajectories for y^{-3} laws.

7.2 Rectilinear motion

We now consider the case of rectilinear motion. This was treated by Newton in Section VIII of his Principia. Interestingly, he did not deal with it directly but considered it as a limiting case of planar motion under a central force. We shall do the same here, with the difference that we shall employ a parallel force.

Firstly we note that this is equivalent to solving a differential equation of the form

$$y'' = F(y)$$

for the special case where F is a power function. Now there is a standard method for solving equations of the above type by quadrature: we introduce the variable w where $y' = w(y)$. Then $y'' = w \frac{dw}{dy}$, so the equation reduces to $\frac{d}{dy} w^2 = 2F(y)$ and so we have $w^2 = 2F_1(y) + C$, i.e., $w = \sqrt{2F_1(y) + C}$ where F_1 is a primitive of F .

Thus we have reduced to the equation $y' = \sqrt{2F_1(y) + C}$, which we can solve by quadrature. (This is essentially the modern version of Newton's treatment as expounded by Chandrasekhar [Ch]). Using the methods of the first section, we can give a more direct approach which does not require quadratures.

If we suppose that f is a function which satisfies the condition

$$f f'' - f'^2 = a f^\alpha$$

and we set $y(t) = f \circ F^{-1}(t)$, (with F again a primitive of f), then the computations above show that

$$\frac{d^2 y}{dt^2} = a y^{-3}$$

and the same argument as we used above shows that we obtain all of the solutions in this manner. Once again, we can choose f as one of the special cases above and thus obtain analytic expressions for rectilinear motion under a parallel power law.

The case of a $\frac{1}{y}$ law is again an exception.

References

[Ar] Arnold V.I., Huygens and Barrow, Newton and Hooke, Berlin 1990.

- [Bi] Biermann, P., Simmons, H.A., A calculus of variations problem whose extremals are parabolas, *Amer. Math. Monthly* 38 (1931) 67-82
- [Br] Brackenridge J. Bruce, The critical role of curvature in Newton's developing dynamics, in *The investigation of difficult things*, 231-260 (CUP, Cambridge, 1992)
- [Ch] Chandrasekhar, S., *Newton's Principia for the common reader* (Oxford, 1975).
- [Co1] Cooper, J.B., Russell, T., Samuelson, P.A., Characterising an area condition associated with minimizing systems, in *Economic Theory, Dynamics and Markets: Essays in honour of Ryuzo Sato* (Editors: T. Nageshi, R.V. Ramachandran and K. Mino), Kluwer, 391-403.
- [Co2] Costabel P., Courbure et dynamique: Jean I. Bernoulli, Correcteur de Huygens et de Newton, *Studia Leibniziana Sonderheft* 17 (1989) 12-24.
- [Go] Gomes Texeira, F. *Traité des courbes* (reprinted, New York, 1971)
- [Ha] Hamilton W.R., The hodograph or a new method of expressing in symbolic language the Newtonian law of attraction, *Proc. Royal Irish Acad.* 3 (1846), 344-353.
- [Ka] Kasner, E., *Differential geometric aspects of dynamics* (American Mathematical Society, New York, 1934).
- [Lo] Loria, G., *Spezielle algebraische und transzendente ebene Kurven*.
- [Ma] MacLaurin, C., *Geometria organica* (London, 1720).
- [Ne] Newton I., *Principia* (Eds. and translators Cohen, I.B., Whiteman, A., Berkeley, 1999).
- [Sc1] Scheffers, G., *Besondere transcendente Kurven* (*Enzyklopaedie der Mathematischen Wissenschaften*, IIID 4).
- [Sc2] Schirakow, P. and A., *Affine Differentialgeometrie* (Leipzig, 1962).
- [Si] Singer D.A., Curves whose curvature depends on distance from the origin, *Amer. Math. Monthly*, 106, (1999), 833-838.
- [Wh] Whittaker, E.T., *A treatise on analytical dynamics of particles and rigid bodies* (Cambridge reprint, 1988).